

# Effective stability of the Trojan asteroids

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**Abstract.** We study the spatial circular restricted problem of three bodies in the light of Nekhoroshev theory of stability over large time intervals. We consider in particular the Sun-Jupiter model and the Trojan asteroids in the neighborhood of the Lagrangian point  $L_4$ . We find a region of effective stability around the point  $L_4$  such that if the initial point of an orbit is inside this region the orbit is confined in a slightly larger neighborhood of the equilibrium (in phase space) for a very long time interval. By combining analytical methods and numerical approximations we are able to prove that stability over the age of the universe is guaranteed on a realistic region, big enough to include one real asteroid. By comparing this result with the one obtained for the planar problem we see that the regions of stability in the two cases are of the same magnitude.

**Key words.** minor planets, asteroids – celestial mechanics – instabilities

## 1. Introduction

The study of a Hamiltonian system in the neighborhood of an elliptic equilibrium point is of interest in many fields of mathematical physics and astronomy. Let us consider an analytic Hamiltonian  $H$  with  $n$  degrees of freedom, having an elliptic equilibrium point. Rigorous results proving the existence of orbits which do not leave a neighborhood of the equilibrium can be given in the frame of KAM theory, under generic conditions of non-resonance and non-degeneracy. KAM guarantees the existence of many  $n$ -dimensional invariant tori around the equilibrium point. However, such invariant tori do not fill an open region, i.e. the possibility of the so-called Arnold diffusion cannot be excluded, except for the two dimensional case.

An alternative approach is to look for results which are valid over a finite time interval, but give an effective bound on the Arnold diffusion. This goal can be achieved by constructing the normal form of the Hamiltonian around the elliptic equilibrium point. Normal forms are a standard tool in Celestial Mechanics for studying the dynamics in the neighborhood of an elliptic equilibrium point. Usually these normal forms are obtained as divergent series but their truncation makes them useful. Roughly speaking one shows that the system admits a number of approximate integrals, whose time variation can be controlled to be small for an extremely long time. In these cases we have effective stability, i.e. even when an orbit is not stable, the time needed for it to leave the neighborhood of the equilibrium is larger than the expected lifetime of

the physical system studied. This is the basis to derive the classical Nekhoroshev's estimates (Nekhoroshev 1977). Different proofs of the Nekhoroshev theorem were given by Benettin et al. (1985), Benettin & Gallavotti (1986), Giorgilli & Zehnder (1992) and Pöschel (1993).

A scientific field where the Nekhoroshev theory has been applied is the problem of the stability of the Trojan asteroids. In recent years this problem has been investigated by a number of researchers, both numerically and analytically. The numerical investigations deal with the evolution in time of a sample of orbits, in sophisticated realistic models of the solar system, and the statistical study of these orbits (Milani 1993, 1994; Levison et al. 1997; Tsiganis et al. 2000).

In analytical studies simpler models of the system have been used such as the two dimensional (2D) planar, and the three dimensional (3D) spatial restricted three body problem. According to Nekhoroshev theory, one has to estimate the rate of diffusion around the elliptic equilibrium Lagrangian point  $L_4$ . Because of the symmetries of the system the same study is valid for the  $L_5$  point. The problem has been previously investigated by Giorgilli et al. (1989), Celletti & Giorgilli (1991) and Giorgilli & Skokos (1997) (hereafter Paper I). Also analytical stability results for particular orbits were provided by Celletti & Ferrara (1996) who studied the orbit of the asteroid Ceres and by Jorba & Villanueva (1998) who studied a stable periodic orbit around the equilibrium point  $L_5$ .

The estimation of the region of effective stability by Giorgilli et al. (1989) and Celletti & Giorgilli (1991) was realistic but the region where the real asteroids were actually found was larger by a factor 300 (in the best case)

to 3000, compared to the theoretical stability region. The theoretical estimation was significantly improved in Paper I, since the region found in the planar restricted three body problem was big enough to include 4 real asteroids, while most of them failed to be inside this region by a factor only 10. This result underlined the fact that Nekhoroshev theory can give meaningful estimates, applicable to real systems. On the other hand Arnold diffusion, which, as already mentioned, can drive some orbits with initial conditions near the equilibrium point  $L_4$  to regions of the phase space far away from it, appears only in the spatial case and not in the planar one.

So in the present paper we study the spatial case, following a similar procedure to the one used in Paper I, in order to find out to what degree the presence of Arnold diffusion changes the effective stability region. Apart from the fact that we consider the three dimensional case and not its restriction on the plane, some minor changes of the scheme used in Paper I improve slightly the estimation of the size of the stability region. In particular the expansion of the Hamiltonian of the system in a power series suitable for the application of the normal form scheme is computed with higher accuracy than before. Also a more accurate calculation of the time needed for an orbit to leave a particular region of the phase space around the point  $L_4$  (escape time) is provided. We were also able to compute the normal form to higher orders than in previous studies, both in the 3D and the 2D case. In particular in the 3D case we have numerically computed the normal form up to order 29, which is a hard task, since one has to manipulate functions with a much larger number of coefficients, than in the case of order 22 (Celletti & Giorgilli 1991). In the 2D case we computed the normal form up to order 49 (instead of order 34 in Paper I).

The paper is organized as follows. In Sect. 2 we present the Hamiltonian we use, referring to all the canonical transformations needed to bring it in a form suitable for the application of the normal form scheme. Also in this section we sketch the procedure of computing the normal form. The main results of the paper, concerning the estimation of the size of the regions of effective stability both in the spatial (3D) and the planar (2D) case, are presented in Sect. 3. The application of the above results to real asteroids is also included in Sect. 3. In Sect. 4 we discuss some issues concerning the effectiveness of our scheme. Finally in Sect. 5 we summarize our results.

## 2. The Hamiltonian and the normal form of the system

The spatial restricted problem of three bodies, in particular for the Sun (S), Jupiter (J) and asteroid (A) system can be described as follows: we study the motion of an asteroid A of infinitesimal mass, orbiting in the gravitational field of two primaries S and J with masses equal to  $1 - \mu$  and  $\mu$  respectively, which are assumed to revolve in circular orbits around their common center of mass.

We introduce a uniformly rotating frame (O,  $q_1, q_2, q_3$ ) so that its origin is located at the center of mass of the Sun-Jupiter system, with the Sun always at the point  $(\mu, 0, 0)$  and Jupiter at the point  $(1 - \mu, 0, 0)$ . The physical units are chosen so that the distance between Jupiter and the Sun is 1,  $\mu = 9.5387536 \cdot 10^{-4}$  and the angular velocity of Jupiter is 1. The time unit is  $(2\pi)^{-1} T_J$ , where  $T_J$  is the period of the circular motion of Jupiter around the Sun. So the age of the universe is about  $10^{10}$  time units. The Hamiltonian of the system is:

$$H = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + q_2 p_1 - q_1 p_2 - \frac{1 - \mu}{\sqrt{(q_1 - \mu)^2 + q_2^2 + q_3^2}} - \frac{\mu}{\sqrt{(q_1 + 1 - \mu)^2 + q_2^2 + q_3^2}}. \quad (1)$$

The coordinates of the Lagrangian point  $L_4$  are:  $q_1 = \mu - \frac{1}{2}$ ,  $q_2 = \frac{\sqrt{3}}{2}$ ,  $q_3 = 0$ ,  $p_1 = -\frac{\sqrt{3}}{2}$ ,  $p_2 = \mu - \frac{1}{2}$ ,  $p_3 = 0$ .

In order to bring the Hamiltonian to a form suitable for the application of the normal form scheme we perform a sequence of transformations. The first step is to introduce a uniformly rotating frame with its origin on the Sun (S) using the generating function

$$W_3 = -(Q_1 + \mu)p_1 - Q_2 p_2 - Q_3 p_3 + \mu Q_2,$$

where  $Q_1, Q_2, Q_3, P_1, P_2, P_3$  are the heliocentric coordinates.

It is known that the projection on the plane of Jupiter's orbit, of the stability region around  $L_4$ , is a banana-shaped region which lies close to the circle with center the Sun and radius equal to the Sun-Jupiter distance. Since the plane of Jupiter's orbit is a symmetry plane for the system, a good choice for describing this region are the cylindrical coordinates  $P, \Theta, Z$ , which are introduced by the generating function

$$W_3 = -P(P_1 \cos \Theta + P_2 \sin \Theta) - Z P_3.$$

In this system of coordinates  $L_4$  is located at  $P = 1$ ,  $\Theta = \frac{2\pi}{3}$ ,  $Z = 0$ ,  $p_P = 0$ ,  $p_\Theta = 1$ ,  $p_Z = 0$ .

We move the origin of the coordinate system to the point  $L_4$  using the generating function

$$W_2 = p_x(P - 1) + (p_y + 1)\Theta - \frac{2\pi p_y}{3} + p_z Z.$$

Then the Hamiltonian becomes:

$$H = \frac{1}{2} \left[ p_x^2 + \frac{(p_y + 1)^2}{(x + 1)^2} + p_z^2 \right] - p_y - \mu(x + 1) \cos \left( y + \frac{2\pi}{3} \right) - \frac{1 - \mu}{\sqrt{(x + 1)^2 + z^2}} - \frac{\mu}{\sqrt{(x + 1)^2 + z^2 + 1 + 2(x + 1) \cos \left( y + \frac{2\pi}{3} \right)}}. \quad (2)$$

We expand the above Hamiltonian in Taylor series around the point  $L_4$  ( $x = y = z = p_x = p_y = p_z = 0$ ) using

the computer algebra platform “Mathematica” (Wolfram Research Inc.). The program allows us to compute the coefficients of the expansion using arbitrary precision arithmetics, while in Paper I the corresponding expansion was made with less accuracy. This change improves the credibility of our computation. After the Taylor expansion the Hamiltonian becomes  $H = \sum_{s=2} H_s$  where  $H_s$  is a homogeneous polynomial of degree  $s$  in  $x, y, z, p_x, p_y, p_z$ . The second order term is

$$H_2 = \frac{1}{2}(p_x^2 + p_y^2) - 2xp_y + \left(\frac{1}{2} + \frac{9\mu}{8}\right)x^2 - \frac{9\mu}{8}y^2 + \frac{3\sqrt{3}\mu}{4}xy + \frac{1}{2}(p_z^2 + z^2). \quad (3)$$

The last change of variables

$$(x, y, z, p_x, p_y, p_z) \rightarrow (x_1, x_2, x_3, y_1, y_2, y_3) \quad (4)$$

is performed in order to bring the quadratic part  $H_2$  (Eq. (3)) to the diagonal form

$$H_2 = \frac{1}{2} \sum_{j=1}^3 \omega_j (x_j^2 + y_j^2). \quad (5)$$

Since the term  $\frac{1}{2}(p_z^2 + z^2)$  in Eq. (3) has this form with  $\omega_3 = 1$ , the canonical transformation (Eq. (4)) must bring to diagonal form the part of  $H_2$  which corresponds to the planar case and depends only on  $x, y, p_x, p_y$ . This transformation is the one performed in Paper I, but we include it to make the paper self-consistent. So we have

$$(x, y, p_x, p_y)^T = C(x_1, x_2, y_1, y_2)^T, \quad (6)$$

where  $T$  denotes the transpose matrix and  $C$  is the matrix

$$C = (e_1 m_1^{-\frac{1}{2}}, e_2 m_2^{-\frac{1}{2}}, f_1 m_1^{-\frac{1}{2}}, f_2 m_2^{-\frac{1}{2}}) \quad (7)$$

with

$$e_j = \left( \frac{8\omega_j^2 + 4\sqrt{3}\alpha + 9}{8}, \frac{4\alpha + 3\sqrt{3}}{8}, 0, \frac{4\sqrt{3}\alpha + 9}{4} \right)^T,$$

$$f_j = \left( 0, 2\omega_j, \frac{\omega_j(8\omega_j^2 + 4\sqrt{3}\alpha + 9)}{8}, \frac{\omega_j(4\alpha + 3\sqrt{3})}{8} \right)^T,$$

$$\frac{m_j}{\omega_j} = \left( \frac{8\omega_j^2 + 4\sqrt{3}\alpha + 9}{8} \right)^2 - 2 \left( \sqrt{3}\alpha + \frac{9}{4} \right) + \left( \frac{4\alpha + 3\sqrt{3}}{8} \right)^2,$$

for  $j = 1, 2$  and

$$\omega_1 = \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{27}{4} + 4\alpha^2}},$$

$$\omega_2 = -\sqrt{\frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{27}{4} + 4\alpha^2}},$$

$$\alpha = -\frac{(1 - 2\mu)3\sqrt{3}}{4}.$$

We remark that the transformation of Eq. (6) involves only the variables  $x, y, p_x, p_y$  since  $z, p_z$  remain unchanged. For notational consistency we complete the transformation of Eq. (6) by putting  $x_3 = z$  and  $y_3 = p_z$ .

All the above transformations were performed in order to bring the Hamiltonian to the form

$$H(x_1, x_2, x_3, y_1, y_2, y_3) = \sum_{s \geq 2} H_s(x_1, x_2, x_3, y_1, y_2, y_3), \quad (8)$$

where  $H_s$  is a homogeneous polynomial of degree  $s$  in the hereafter called “old variables”  $x_1, x_2, x_3, y_1, y_2, y_3$ . The quadratic part  $H_2$  which is given by Eq. (5) is the Hamiltonian of a system of three harmonic oscillators with frequencies  $\omega_1 \simeq 9.967575 \cdot 10^{-1}$ ,  $\omega_2 \simeq -8.046388 \cdot 10^{-2}$  and  $\omega_3 = 1$ . The form of Eq. (8) is suitable for the direct application of the normal form theory, as it is described in detail by Giorgilli et al. (1989). We give a brief sketch of this procedure: we construct a generating function  $X$ , of the form

$$X = \sum_{s \geq 3} X_s, \quad (9)$$

where  $X_s$  is a homogeneous polynomial of degree  $s$ , so that the corresponding canonical transformation brings the Hamiltonian to normal form

$$Z = \sum_{s \geq 2} Z_s, \quad (10)$$

where  $Z_s$  is a homogeneous polynomial of degree  $s$  in the new “normal variables”  $x'_1, x'_2, x'_3, y'_1, y'_2, y'_3$ . The term “normal form” means, that  $Z$  is a function of the quantities  $I'_i = \frac{1}{2}(x'_i{}^2 + y'_i{}^2)$  with  $i = 1, 2, 3$ , so that the system is formally integrable (Birkhoff 1927; Gustavson 1966).

The generating function  $X$  and the normal form  $Z$  are computed by solving the equation

$$T_X Z = H, \quad (11)$$

where  $T_X$  is an operator whose action is defined as follows

$$T_X Z = \sum_{k \geq 1} F_k,$$

where

$$F_k = \sum_{s=1}^k Z_{s,k-s},$$

$$Z_{s,0} = Z_s, \quad Z_{s,k} = \sum_{n=1}^k \frac{n}{k} \cdot L_{X_{2+n}} Z_{s,k-n}$$

and  $L_{X_k} Z_s = [X_k, Z_s]$ , with  $[,]$  denoting the Poisson bracket. The operator  $T_X$  is linear, invertible and preserves products and Poisson brackets (Giorgilli & Galgani 1978). The computer program that solves Eq. (11) and determines the generating function  $X$  and the normal form  $Z$  is described by Giorgilli (1979).

Since the Hamiltonian in the old variables (Eq. (8)) is an infinite series, in practice we stop the expansion at some order  $\tilde{r}$  and use the truncated Hamiltonian

$$H^{(\tilde{r})} = H_2 + H_3 + \dots + H_{\tilde{r}}. \quad (12)$$

Then for any fixed integer  $r$  with  $3 \leq r < \tilde{r}$  we solve Eq. (11) defining a truncated generating function  $X^{(r)}$  up to order  $r$

$$X^{(r)} = X_3 + X_4 + \dots + X_r \quad (13)$$

and constructing the normal form  $Z^{(r)}$  up to this order

$$Z^{(r)} = Z_2 + Z_3 + \dots + Z_r + Y^{(r)}, \quad (14)$$

where  $Y^{(r)}$  is a remainder, actually a power series starting with terms of degree  $r + 1$ . So we have the equation

$$T_{X^{(r)}}^{-1} H^{(r)} = \underbrace{Z_2 + Z_3 + \dots + Z_r}_{\text{normal form}} + \underbrace{Y^{(r)}}_{\text{remainder}}. \quad (15)$$

The first term of the remainder  $Y_{r+1}^{(r)}$  is also computed, since it is needed for the estimation of the size of the effective stability region as we will explain in Sect. 3.

In Paper I, where the planar problem (2D case) was studied, the power series of 4 variables were truncated at order  $\tilde{r} = 35$ . A function of 4 variables expanded up to order 35 requires 82 251 coefficients, while the process of constructing the normal form requires the computation of several functions with a total of 2 549 782 coefficients. In the spatial problem (3D case) we use expansions of functions of 6 variables up to order  $\tilde{r} = 30$ . This is a much harder task compared to the 2D case since a function of 6 variables expanded up to order 30 requires 1 947 792 coefficients and the program which calculates the normal form manipulates 55 929 459 coefficients.

### 3. Estimation of the effective stability region

#### 3.1. Theoretical framework

The transformed Hamiltonian  $Z^{(r)}$  admits three approximate first integrals of the form

$$I_j'(x', y') = \frac{1}{2} (x_j'^2 + y_j'^2), \quad j = 1, 2, 3. \quad (16)$$

The variation of these quantities in time, is given by

$$\dot{I}_j' = [I_j', Z^{(r)}] = [I_j', Y^{(r)}], \quad j = 1, 2, 3, \quad (17)$$

which is a power series starting with terms of degree  $r + 1$ .

We introduce now suitable domains in the phase space, where we study the stability properties of the system and also a norm in these domains, in order to estimate the time variations of the three approximate integrals  $\dot{I}_j'$ .

For arbitrary fixed positive constants  $R_1, R_2, R_3$  we consider a family of domains of the form

$$\Delta_{\rho R} = \left\{ (x', y') \in \mathbf{R}^6 : x_j'^2 + y_j'^2 \leq \rho^2 R_j^2 \right\}, \quad (18)$$

where  $\rho$  is a positive parameter and  $x', y'$  stand for  $x_1', x_2', x_3'$  and  $y_1', y_2', y_3'$  respectively. From the definition of the domains  $\Delta_{\rho R}$  it is evident that

$$(x', y') \in \Delta_{\rho R} \Rightarrow I_j' \leq \frac{1}{2} \rho^2 R_j^2, \quad j = 1, 2, 3. \quad (19)$$

The norm  $\|f\|_{\rho R}$  of a homogeneous polynomial  $f(x', y')$  of degree  $s$  in the domain  $\Delta_{\rho R}$  does not exceed the quantity

$$\|f\|_{\rho R} \leq \frac{\rho^s}{2^{s/2}} \sum_{j_1 j_2 j_3 k_1 k_2 k_3} |C_{j_1 j_2 j_3 k_1 k_2 k_3}| R_1^{j_1+k_1} R_2^{j_2+k_2} R_3^{j_3+k_3} \quad (20)$$

as shown in Paper I.  $C_{j_1 j_2 j_3 k_1 k_2 k_3}$  are the complex coefficients of  $f(x', y')$  when  $f$  is transformed in complex variables  $\xi, \eta$  via the transformation  $x_j' = (\xi_j + i\eta_j)/\sqrt{2}$ ,  $y_j' = i(\xi_j - i\eta_j)/\sqrt{2}$  for  $j = 1, 2, 3$ . We remark that for the above norm the elementary property

$$\|f\|_{\rho R} = \rho^s \|f\|_R \quad (21)$$

holds.

We assume that the initial point of an orbit lies in the domain  $\Delta_{\rho_0 R}$  for some positive value  $\rho_0$ , and we require that the orbit should be confined inside a domain  $\Delta_{\rho R}$  with  $\rho > \rho_0$  for a finite time interval. We shall refer to this time interval as the escape time  $\tau$ . Since  $\dot{I}_j' = dI_j'/dt$ , we get

$$dt \geq \frac{dI_j'}{\sup_{\Delta_{\rho R}} |\dot{I}_j'|}, \quad j = 1, 2, 3, \quad (22)$$

where  $\sup_{\Delta_{\rho R}} |\dot{I}_j'|$  denotes the maximum absolute value of  $\dot{I}_j'$ , or in other words the supremum norm of  $\dot{I}_j'$ , over the domain  $\Delta_{\rho R}$ .

Giorgilli et al. (1989) proved that the power series of the remainder  $Y^{(r)}$  is absolutely convergent in a domain  $\Delta_{\rho R}$  provided  $\rho$  is small enough. So, assuming that  $\rho$  is smaller than half of the convergence radius of the remainder  $Y^{(r)}$  we can use the approximate estimation

$$\sup_{\Delta_{\rho R}} |\dot{I}_j'| < 2 \|[I_j', Y_{r+1}^{(r)}]\|_{\rho R} \stackrel{(21)}{=} 2\rho^{r+1} \|[I_j', Y_{r+1}^{(r)}]\|_R, \quad (23)$$

where  $Y_{r+1}^{(r)}$  is the first term of the remainder. The term  $Y_{r+1}^{(r)}$  is a homogeneous polynomial of degree  $r + 1$  and it is easily computed as a byproduct of the program that calculates the normal form. The validity of the above assumption is discussed in Sect. 4.

By integrating both parts of Eq. (22) and using Eq. (23), we estimate the minimum escape time as

$$\tau_r(\rho_0, \rho) = \min_{j=1, 2, 3} \frac{R_j^2}{2(r-1) \|[I_j', Y_{r+1}^{(r)}]\|_R} \left[ \frac{1}{\rho_0^{r-1}} - \frac{1}{\rho^{r-1}} \right]. \quad (24)$$

The above quantity depends on the order  $r$  up to which the normal form is constructed and on the radii  $\rho_0$  and  $\rho$  of the initial and final domain respectively. In Eq. (24)

we keep the smallest value with respect to  $j$  ( $j = 1, 2, 3$ ), because when the orbit is outside the disk with radius  $\rho R_j$  in one of the planes  $(x'_j, y'_j)$  it is also outside  $\Delta_{\rho R}$ .

In order to have the minimum escape time as a function of  $\rho_0$  we eliminate the dependence of  $\tau_r(\rho_0, \rho)$  on  $\rho$  and  $r$ . It is evident that for a given order  $r$  the escape time goes to infinity as the radius of the outer domain grows. So by fixing  $\rho$  to be equal to  $\lambda\rho_0$ , with  $\lambda > 1$  we get

$$\tau_{r,\lambda}(\rho_0) = \min_{j=1, 2, 3} \frac{R_j^2}{2(r-1)\rho_0^{r-1} \| [I'_j, Y_{r+1}^{(r)}] \|_R} \left[ 1 - \frac{1}{\lambda^{r-1}} \right]. \quad (25)$$

By giving a fixed value to  $\lambda$  we eliminate the dependence of the minimum escape time on the radius of the final domain. In particular, we put  $\lambda = 1.2$ , which means that the radius of the final domain is 20% greater than the radius of the initial domain.

The next step is to optimize the minimum escape time with respect to  $r$ . For  $\lambda = 1.2$  we compute  $\tau_{r,1.2}(\rho_0)$  via Eq. (25) for  $r$  running from 3 to the maximum order  $\tilde{r} - 1$ , for every value of  $\rho_0$ . We choose the optimal order  $r_{\text{opt}}$  of the expansion as the one that gives the maximum value of the escape time. Thus we get the maximum escape time  $T$  as function of only the radius  $\rho_0$  of the initial domain:

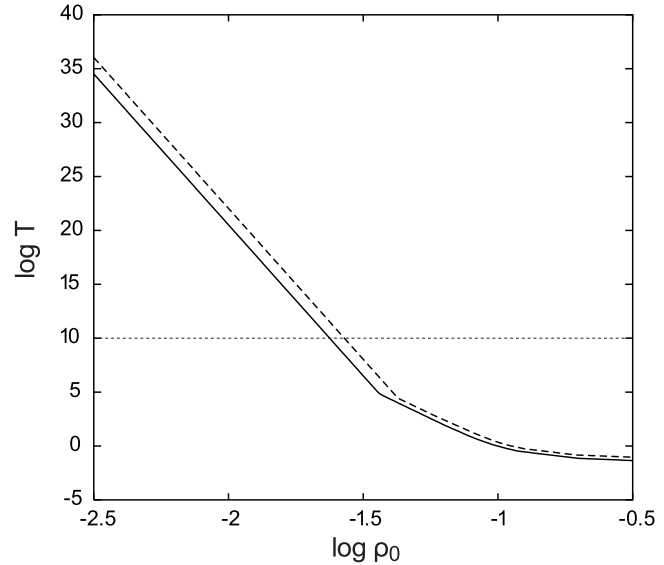
$$T(\rho_0) = \max_{3 \leq r < \tilde{r}} \tau_{r,1.2}(\rho_0). \quad (26)$$

The maximum order of the power expansions in the 3D case is  $\tilde{r} = 30$ , which means that the normal form was computed up to order 29 and the first term of the remainder is of order 30. So it becomes clear that  $r_{\text{opt}} \leq 29$ .

### 3.2. General results

For a general discussion and for making the results comparable to the ones found in Paper I for the 2D case, we put  $R_1 = R_2 = R_3 = 1$ . In Fig. 1 we plot the logarithm of the maximum escape time ( $\log T$ ) as a function of the logarithm of the radius of the initial domain ( $\log \rho_0$ ), in the spatial (3D) and the planar (2D) cases. In both cases the normal form is computed up to order 29. Taking an initially small domain around the  $L_4$  point, all the orbits are confined inside a slightly larger domain (with 20% greater radius), for very long time intervals, while for large initial domains (large values of  $\rho_0$ ), the escape time is small.

A meaningful time interval for the system is the estimated age of the universe, which is in our time units  $10^{10}$ . This value is marked in Fig. 1 by a horizontal line and it corresponds to  $\log \rho_0 \simeq -1.625$ , i.e.  $\rho_0 \simeq 2.371 \cdot 10^{-2}$  in the 3D case and to  $\log \rho_0 \simeq -1.570$ , i.e.  $\rho_0 \simeq 2.692 \cdot 10^{-2}$  in the 2D case. In both cases the optimal order of the expansion is  $r_{\text{opt}} = 29$ . The best previously found estimation of the radius of the effective stability region, was obtained in Paper I in the 2D case, namely  $\rho_0 \simeq 2.911 \cdot 10^{-2}$ , where the normal form was computed up to order 34. In that case the optimal order was  $r_{\text{opt}} = 34$ . The radius of the effective stability region in the spatial case is about 12% smaller



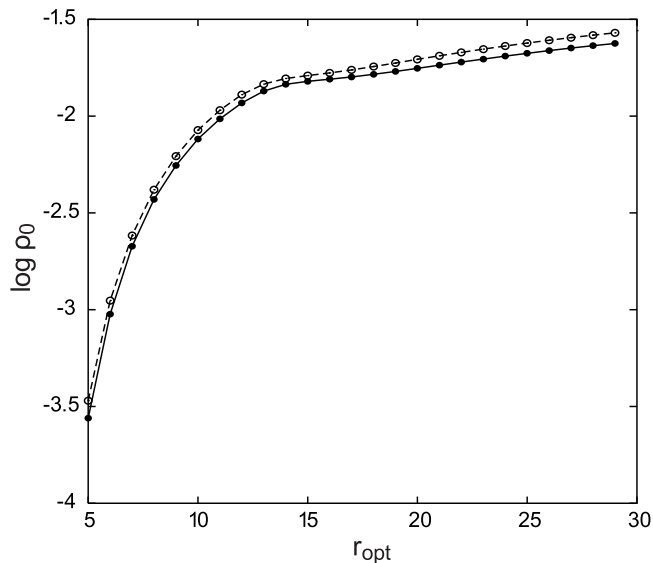
**Fig. 1.** The logarithm of the maximum escape time  $\log T$  as a function of the logarithm of the radius of the initial domain  $\log \rho_0$ , in the spatial 3D case (solid line) and in the planar 2D case (dashed line). In both cases the normal form has been computed up to order 29. The horizontal line marks the time that corresponds to the age of the universe

than the one computed for the planar case for  $r_{\text{opt}} = 29$  and about 19% smaller than the one found in Paper I for  $r_{\text{opt}} = 34$ . So, the estimated stability region in the 3D case is a realistic one since it is comparable to the region found in Paper I.

The estimated radii in the 3D and the 2D cases, for the same order of expansion of the normal form, are relatively close to each other since they are of the same order of magnitude, with the radius computed in the 3D case being always smaller, as seen in Fig. 2. Thus the Arnold diffusion, which appears only in the 3D case, does not affect the size of the effective stability region significantly.

We notice in Fig. 2 that up to order  $r_{\text{opt}} = 14$  the increment of the order  $r_{\text{opt}}$  improves the estimation of the radius significantly both in the 3D and the 2D case. For orders greater than 15 the increment of the order leads to big increment of the computational effort needed for the construction of the normal form (mainly in the 3D case), but to relatively small improvements for the estimated radii. For instance, the radius found in the 2D case for  $r_{\text{opt}} = 13$  is almost equal to the one found for  $r_{\text{opt}} = 14$  in the 3D case ( $\log \rho_0 \simeq -1.84$ ) which was obtained by computing almost 20 times more terms for the various expansions, than in the 2D case. As  $r_{\text{opt}}$  increases things become even more difficult. For  $r_{\text{opt}} = 29$  in the 3D case we find almost the same radius as in the 2D case for  $r_{\text{opt}} = 25$  by computing almost 86 times more terms compared to the 2D case. So, it is evident that pushing the computation of the normal form to higher orders becomes impractical for the 3D case. On the other hand this can be done more easily in the 2D case.

According to Nekhoroshev theory, the series arising from the classical perturbation theory have an asymptotic character in the sense that one should not exceed an



**Fig. 2.** The logarithm of the radius  $\log \rho_0$  of the effective stability region which ensures stability for time equal to the age of the universe, as a function of the optimal order  $r_{\text{opt}}$  of the expansion of the normal form, in the spatial 3D case (solid line) and in the planar 2D case (dashed line). In both cases the normal form has been computed up to order 29

optimal value for the order of expansion, which gives the best possible result. From Fig. 2 we see that this limit is greater than 30, both for the 3D and the 2D case. In Paper I, where the construction of the normal form was done up to order 34 for the 2D case, the limit  $r_{\text{opt}}$  was not reached.

By computing the normal form up to order 49 for the planar problem, so that the first term of the remainder is of order 50, we find the maximum value of the optimal order of the expansion of the normal form to be  $r_{\text{opt}} = 38$ , and the corresponding value of  $\log \rho_0$  equal to  $-1.506$ , i.e.  $\rho_0 \simeq 3.119 \cdot 10^{-2}$ . This value is about 7% greater than the one found in Paper I for  $r_{\text{opt}} = 34$ . This is the best estimation of the radius of the stability region one can achieve using the particular theoretical framework, since expansions of the normal form to orders greater than 38 do not improve the results.

### 3.3. Application to real asteroids

In order to apply the above results to the real solar system we examine if 98 real asteroids, which are located near the Lagrangian point  $L_4$ , are inside the estimated effective stability region. Using the same catalog as in Paper I we extract the elements of the Trojan asteroids at the epoch, December 14, 1994,  $\text{JD} = 2449700.5$ . We also find the elements of Jupiter at the same epoch. Assuming that the orbit of Jupiter is circular we find the position of the  $L_4$  point and compute the coordinates  $(Q_1, Q_2, Q_3, P_1, P_2, P_3)$  of the asteroids in the rotating heliocentric system with the  $z$  axis orthogonal to the plane of Jupiter's orbit. By using the canonical transformations described in Sect. 2 we find the position of every asteroid in the  $(x_1, x_2, x_3, y_1, y_2, y_3)$  coordinates which diagonalize the quadratic part (Eq. (3)) of the Hamiltonian.

Then we define the radii  $R_1, R_2, R_3$  for every asteroid as  $R_j = \sqrt{x_j^2 + y_j^2}$  for  $j = 1, 2, 3$ . Using these values we determine the radius  $\rho_0$  of the effective stability region in the way described previously in this section. It is evident from the definition of  $R_1, R_2, R_3$ , that an asteroid is inside the stability region if  $\rho_0 \geq 1$ .

In the above procedure we use exactly the same asteroids and their elements at the same epoch, as in Paper I. In this way our results can be compared directly to the ones obtained in Paper I where the effective stability of 4 real asteroids was guaranteed. We apply this procedure both to the spatial case, where the normal form is computed up to order 29 and to the planar case, where the normal form is computed up to order 49. We remark that in order to find the radius  $\rho_0$  in the 2D case only the values of  $R_1$  and  $R_2$  are used, since we project the real asteroids on the plane of Jupiter's orbit.

In Table 1 only the results for the 5 best and the 5 worst cases of the radius  $\rho_0$  in the 3D case are presented. For every asteroid the corresponding results in the 2D case are also reported. In the 3D case one real asteroid is inside the stability region, while in the worst case the estimated value of the stability region's radius is smaller by a factor 34. In all cases the optimal order of the normal form is the maximum possible,  $r_{\text{opt}} = 29$ , which means that the results may be improved for higher orders.

In the 2D case the results are slightly better than the ones achieved in Paper I, mainly because we performed the expansion of the normal form up to a higher order. This improvement does not change the results significantly. Four real asteroids (three reported in Table 1 and one not shown in that table) are inside the planar stability region. In the worst case for the planar problem (asteroid 2363) a factor 27 is needed in order for the asteroid to be safely inside the stability region. The optimal order for all asteroids is  $r_{\text{opt}} \leq 38$ , although the expansion of the normal form was performed up to order 49. So the computation of the normal form to orders higher than 38 does not improve the estimations in the 2D case.

We remark that one would expect to find fewer asteroids inside the stability region in the 3D case than in the 2D case, since the spatial stability region is projected on a plane in the 2D case. So points that are outside the spatial stability region may be projected inside the planar stability region.

## 4. Discussion

In Sect. 3.1 we assumed that the supremum of  $|I_j'|$ , which is an infinite series, can be approximated by the norm of the first term multiplied by 2 (Eq. (23)). The practical reason for doing this approximation is that we cannot compute the whole infinite series, while on the other hand its first term can be obtained easily. The above assumption for the norm is valid if the estimated value  $\rho_0 \simeq 2.371 \cdot 10^{-2}$  in the spatial case, is smaller than the half of the convergence radius of the remainder  $Y^{(r)}$ . But estimating the convergence radius of the remainder is not possible

**Table 1.** Estimated stability region for some real asteroids. CN is the catalog number of the asteroids and  $R_1, R_2, R_3$  are the radii corresponding to the initial data. The radius  $\rho_0$  of the effective stability region is computed in the spatial (3D) case where the normal form was constructed up to order 29 and in the planar (2D) case where the normal form was constructed up to order 49. An asteroid is inside the stability region if  $\rho_0 \geq 1$ . The optimal order of the expansion  $r_{\text{opt}}$  is also reported in both cases. The table is sorted in decreasing order with respect to the value of  $\rho_0$  for the spatial case

CN	$R_1$	$R_2$	$R_3$	3D case		2D case	
				$\rho_0$	$r_{\text{opt}}$	$\rho_0$	$r_{\text{opt}}$
89211605	0.033150	0.019594	0.013270	1.0105	29	1.1722	35
2357	0.042346	0.028509	0.049056	$6.3957 \cdot 10^{-1}$	29	$8.7950 \cdot 10^{-1}$	36
88181612	0.031302	0.002101	0.068011	$6.3294 \cdot 10^{-1}$	28	1.5364	33
41790004	0.016517	0.031063	0.068685	$6.2154 \cdot 10^{-1}$	29	1.1687	36
5257	0.031836	0.042424	0.031626	$6.1978 \cdot 10^{-1}$	29	$8.0918 \cdot 10^{-1}$	38
4867	0.233120	0.736264	0.468336	$4.0589 \cdot 10^{-2}$	29	$5.1663 \cdot 10^{-2}$	36
1867	0.216926	0.758254	0.466126	$3.9819 \cdot 10^{-2}$	29	$5.0305 \cdot 10^{-2}$	36
88172500	0.242971	0.902082	0.524644	$3.3747 \cdot 10^{-2}$	29	$4.2302 \cdot 10^{-2}$	36
2363	0.293736	1.012524	0.553082	$3.0026 \cdot 10^{-2}$	29	$3.7667 \cdot 10^{-2}$	36
1208	0.361925	0.997579	0.543939	$2.9917 \cdot 10^{-2}$	29	$3.7807 \cdot 10^{-2}$	36

since it requires the computation of higher orders of the series.

One can attempt an estimation of the convergence radius following Paper I. Since we have computed the expansion of the generating function  $X$  (Eq. (9)) up to order 30, we can evaluate its convergence radius by fitting the norms  $\|X_s\|_R$  for  $3 \leq s \leq 30$  with a geometric sequence, i.e. we look for constants  $a$  and  $b$  such that

$$\|X_s\|_R \leq ab^{s-3}. \quad (27)$$

In order to satisfy this condition it is sufficient to set

$$a = \|X_3\|_R, \quad b = \max_{3 \leq s \leq 30} \left( \frac{\|X_s\|_R}{\|X_3\|_R} \right)^{\frac{1}{s-3}}. \quad (28)$$

Giorgilli et al. (1989) proved that if the generating function  $X$  satisfies Eq. (27) then the coordinate transformation  $x' = T_X x$ ,  $y' = T_X y$  is absolutely convergent in a domain  $\Delta_{\rho R}$  provided  $\rho$  is small enough. The same was proven for the inverse coordinate transformation and for the transformation of any other function such as the actions  $I'_j$  (Eq. (16)). So, we fit with a geometric sequence the norms of the coordinate transformations  $T_X x_1, T_X x_2, T_X x_3, T_X y_1, T_X y_2, T_X y_3$  and of the transformations of the approximate integrals  $I'_j$  to the old variables  $T_X^{-1} I'_1, T_X^{-1} I'_2, T_X^{-1} I'_3$ . The worst case ( $T_X^{-1} I'_2$ ) gives  $b \simeq 18.665$ , which corresponds to a convergence radius  $\rho \simeq 5.358 \cdot 10^{-2}$ . Considering this value as a good indicator of the true convergence radius, we conclude that the estimated radius of the stability region is smaller than the convergence radius of the series, by a factor 2.2, so it is safely inside the convergence domain and the approximation used in Eq. (23) holds.

The theoretical framework we used proved to be very efficient, since we are able to guarantee the effective stability of one and four real asteroids in the spatial and planar problem, respectively. On the other hand we have reached the limits of its effectiveness, since the above results cannot be improved significantly.

Computing the expansion of the normal form to higher orders in the 3D case would slightly improve the estimations, as we can see from the respective estimations performed in the 2D case, where the optimal order was reached for  $r_{\text{opt}} = 38$ .

The estimation of the minimum escape time  $\tau_r(\rho_o, \rho)$  (Eq. (24)) was improved compared to the one used in Paper I. This improvement was done by taking into account the dependence of  $\sup_{\Delta_{\rho R}} |\dot{I}'_j|$  on the radius  $\rho$  of the final region (Eq. (23)), while in Paper I this quantity had been overestimated by considering  $\sup_{\Delta_{\rho R}} |\dot{I}'_j|$  constant inside the domain  $\Delta_{\rho R}$  and equal to its value at the edge of the domain. The above change forced us to fix the radius  $\rho$  of the final domain. In our calculations we used  $\lambda = \rho/\rho_0 = 1.2$  but this value does not influence strongly the results. This can be easily understood since, the quantity  $[1 - (1/\lambda^{r-1})]$  in Eq. (25) is very close to 1 for  $\lambda > 1.1$  and  $r = 29$  (which is the order of expansion up to which we compute the normal form). The improved estimation of the escape time does not change the results drastically. For instance, in the 2D case for  $R_1 = R_2 = 1$  with the normal form being computed up to order 34, we got  $\rho_0 \simeq 3.005 \cdot 10^{-2}$ , which is about 3% greater than the value  $\rho_0 \simeq 2.911 \cdot 10^{-2}$  obtained in Paper I for the same case.

The norms  $\| [I'_j, Y_{r+1}^{(r)}] \|_R$  for  $j = 1, 2, 3$  in Eq. (25) were estimated using Eq. (20). Since this estimation is purely analytic it is certainly pessimistic. For this reason we made numerical calculations of the norms in order to determine whether the overestimation of the norms influence strongly the estimation of the radius of the stability region. We used two Fortran codes. Algorithm GLOBAL (Boender et al. 1982), which had the best performance in all cases, was used to obtain the norms, and algorithm SIGMA (TOMS 667) (Aluffi-Pentini et al. 1988a,b) was used for verification purposes. GLOBAL is a stochastic algorithm for finding the maximum of a real-valued function. In stochastic methods for optimization, the probability of finding the global maximum approaches unity as

the sample size of the random initial values increases. This algorithm utilizes a combination of sampling, clustering, and local search; it terminates with a range of confidence intervals on the value of the global maximum. SIGMA is a global optimization algorithm, which implements a method founded on the numerical solution of a Cauchy problem for a stochastic differential equation inspired by statistical mechanics. A global maximum of the function is sought by monitoring the values of the function along trajectories generated by a suitable discretization of a first-order stochastic differential equation.

In all cases the numerically found maximum of  $\| [I'_j, Y_{r+1}^{(r)}] \|_R$ ,  $j = 1, 2, 3$  was located at the edge of the domain  $\Delta_R$ . So by limiting our computation on the edge we were able to evaluate the maximum with even greater accuracy. The analytically estimated maximum was greater than the one computed numerically by a factor 28 in the worst case. Using the numerically calculated norms in Eq. (26) we found  $\rho_0 \simeq 2.667 \cdot 10^{-2}$ . This value is only 1.12 times greater than the one computed by using the analytically estimated norms. So the improvement of the size of the stability region is negligible compared to the factor 34 needed for all the real asteroids to be inside the stability region. Thus the estimations based on the analytically found norms are reliable.

## 5. Summary

We studied the spatial and the planar circular restricted problem of three bodies in the spirit of Nekhoroshev theory, considering in particular the problem of the stability of the Trojan asteroids around the Lagrangian point  $L_4$  in the Sun–Jupiter–Asteroid model. By constructing the normal form with terms up to order 29 in the spatial case and up to order 49 in the planar case and using analytical methods and numerical approximations, we made realistic estimations of the size of the effective stability region. Our results can be summarized as follows:

1. The estimated size of the effective stability region in the spatial case is big enough to include 1 real asteroid. The region where the most remote asteroid is located (out of the 98 real asteroids we checked), is larger by a factor 34 compared to the estimated stability region. This result improves significantly older estimates (Giorgilli et al. 1989; Celletti & Giorgilli 1991) where no real asteroid was inside the stability region and a factor 3000 was needed for the most remote asteroid to be inside the stability region;
2. The radii of the effective stability region in the general spatial and planar cases are close to each other for the same order of expansion of the normal form, with the radius computed for the spatial case being always slightly smaller. Thus, Arnold diffusion does not affect the size of the effective stability region significantly;
3. By computing the normal form in the planar case, to higher order than in Paper I, the optimal order of the expansion of the normal form was actually reached and it was found to be  $r_{\text{opt}} = 38$ . Also the estimation of the stability region's size was slightly improved compared to Paper I. Four real asteroids are found to be inside the stability region, as it was found also in Paper I;
4. Several improvements in the computational procedure, compared to previous works, were introduced in the present paper, namely: a) the computation of the normal form to higher orders; b) a more accurate estimation of the minimum escape time  $\tau_r(\rho_0, \rho)$  (Eq. (24)); c) the numerical estimation of the norms  $\| [I'_j, Y_{r+1}^{(r)}] \|_R$  for  $j = 1, 2, 3$ , which also verified the reliability of the analytically calculated norms via Eq. (20) and d) the use of an arbitrary precision computer algebra system for the expansion of the Hamiltonian;
5. The theoretical framework we used reached the limits of its effectiveness by providing the best possible results in the 2D case, and comparable results in the 3D case. In order to have a non negligible improvement of the estimated size of the effective stability region, one has to choose better coordinates than the cylindrical ones used in this paper, in the sense that these coordinates should be more adapted to describe the banana-shaped region of the actual stability region around  $L_4$ .

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